Multiinstanton Ladders in Baryon Number Violating Processes

G. Lazarides[*]

Physics Division,School of Technology,

University of Thessaloniki,

Thessaloniki 540 06, Greece

N.D. Vlachos[†]

Dept. of Theoretical Physics

University of Thessaloniki

Thessaloniki 540 06, Greece

Abstract

We estimate the contribution of a class of multiinstanton ladder graphs to baryon and lepton number violating processes in the standard model. We find that this contribution is negligible and does not alter the high energy behavior of the leading semiclassical approximation.

[*]E-mail lazaride@vergina.eng.auth.gr

[†]E-mail vlachos@olymp.ccf.auth.gr. Work supported in part by E.U. grant Science SC1-0729-92

I. INTRODUCTION

Instanton-induced processes in the standard electroweak theory are known to lead to baryon and lepton number violation. Although 't Hooft [1] showed several years ago that such phenomena are utterly suppressed by the factor $e^{-8\pi^2/g^2}$ (g is the $SU(2)_L$ gauge coupling constant), several authors [2] explored the possibility that this exponential suppression factor can be overcome at high energies by the phase space which corresponds to multiparticle production. The key observation is that the $SU(2)_L$- instanton induces to leading semiclassical approximation effective point-like interactions which involve all the fermionic left-handed $SU(2)_L$-doublets of the theory (four per generation) and any number of Higgs and gauge bosons.

The inclusive cross section of the baryon and lepton number violating two fermion scattering can then be calculated as the imaginary part of the forward $2 \rightarrow 2$ scattering amplitude depicted in fig.1 [3]. As a result, this inclusive cross section appears to grow exponentially with energy and can conceivably become unsuppressed at energies of the order of the sphaleron [4] mass. This leading order behavior may, however, be drastically altered by higher order corrections well before the energy reaches the sphaleron mass and consequently, these phenomena may remain unobservable at all energies. Several authors [5] actually, have suggested that this could be the case if multiinstanton corrections were to be taken into account. Corrections to the $2 \rightarrow 2$ scattering amplitude consisting of linear instanton-antiinstanton chains in alternating order were considered, with particle exchange allowed only between successive instantons and antiinstantons. Dorey and Mattis [6], however, using the valley method [7], pointed out that inclusion of non-nearest neighbor instanton-antiinstanton as well as instanton-instanton and antiinstanton-antiinstanton interactions could render these chain graphs unimportant at the relevant energies. This, however, may not happen if Dorey's result on the non linear $O(3)$ $\sigma-$model applies in the

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realistic case too. The imaginary part of the chain graphs being ultraviolately divergent, requires the introduction of an appropriate cut-off. Finally, these graphs do not include initial state corrections and thus, are not expected to alter the high energy behavior of the leading semiclassical approximation [7], [8].

In this work we choose to deal with a class of ladder graphs shown in fig.2. The imaginary part of these graphs turns out to be finite and can, in principle, be unambiguously calculated. They can, in some sense, be thought as including initial state corrections too since the incoming particles enter in different instanton vertices. In addition, such ladder graphs are known to dominate the high-energy behavior in ordinary field theories. We should emphasize, however, that including the ladder graphs of fig.2 does not solve the problem of initial state corrections. Indeed, the entire picture of separated instanton-antiinstanton chains is an uncontrolled approximation at energies where such chains actually become important.

II. THE LEADING SEMICLASSICAL APPROXIMATION.

Consider the inclusive cross section, σ_{inc}, of the B and L violating reaction

$$q + q \rightarrow (3n_g - 2)\overline{q} + n_g\overline{l} + any \# of\ Higgses, \tag{1}$$

where n_g is the number of fermion generations ($n_g \geq 1$), q and l represent quarks and leptons respectively and we have ignored for simplicity the production of gauge particles. In the leading instanton approximation, σ_{inc} can be determined by first calculating the forward $2 \rightarrow 2$ scattering amplitude in Euclidean space as shown in fig.1. Then, σ_{inc} is given by the imaginary part acquired by this amplitude after rotating the total incoming Euclidean particle momentum $p = p_1 + p_2$ to Minkowski space $(p^2 \rightarrow e^{-\imath\pi} p^2)$. The expression corresponding to the Euclidean space forward scattering amplitude is [2], [3]

$$C \int_0^\infty d\rho^2\, \rho^{2\alpha}\, e^{-\pi^2 v^2 \rho^2} \int_0^\infty d\tilde{\rho}^2\, \tilde{\rho}^{2\alpha}\, e^{-\pi^2 v^2 \tilde{\rho}^2} \int d^4x\, e^{-\imath px}\, F(x^2, \rho^2, \tilde{\rho}^2). \tag{2}$$

C is a constant given by

$$C \propto \frac{1}{\pi^2} (32\pi^2)^{4n_g-1} \left(\frac{8\pi^2}{g^2}\right)^8 \mu^{\frac{43-8n_g}{3}} e^{-\frac{16\pi^2}{g^2(\mu)}}, \quad \alpha = (28n_g + 7)/12, \tag{3}$$

ρ and $\tilde{\rho}$ are the scale sizes of the instanton and the antiinstanton, x_μ is their Euclidean separation, $v = 246 GeV$ the electroweak breaking scale, g the gauge coupling and μ is the renormalization point. The function $F(x^2, \rho^2, \tilde{\rho}^2)$, to leading semiclassical order (or for $x^2 \to \infty$), can be written as

$$F(x^2, \rho^2, \tilde{\rho}^2) \equiv F(x^2) = e^{\kappa/x^2} \frac{1}{(x^2)^n}, \tag{4}$$

where $\kappa = \pi^2 \rho^2 \tilde{\rho}^2 v^2$ and $n = 3(2n_g - 1) \geq 3$. The exponential factor in the formula above, corresponds to the Higgses in the final state of reaction (1), whereas the second factor corresponds to the $4n_g - 2$ fermions which are also being produced. Since we are interested in the high energy behavior of σ_{inc}, we can assume throughout this work that all fermions and Higgs bosons are effectively massless.

We will first consider the integral over the instanton-antiinstanton separation

$$I^0(p^2) \equiv \int d^4x \, e^{-\imath px} F(x^2) \quad . \tag{5}$$

This integral converges at infinity for $n \geq 3$, but diverges badly at $x = 0$. This virulent ultraviolet divergence in Euclidean space is due to the attractive nature of the Coulomb potential $V(x^2) = -\kappa/x^2$ resulting from Higgs particle exchange between the instanton and the antiinstanton and is an artifact of the leading semiclassical approximation. We, thus, define a regularized integral

$$I_\delta^0(p^2) = \int_{x^2 \geq \delta^2} d^4x \, e^{-\imath px} F(x^2) \tag{6}$$

by removing from the range of integration a four-dimensional disc of finite radius $\delta > 0$ centered at the origin. Performing the angular integrations, we obtain

$$I_\delta^0(p^2) = 2\pi^2 \int_\delta^\infty dr \, e^{\kappa/r^2} r^{3-2n} G_{0\,2}^{1\,0}(\frac{p^2 r^2}{4}|0, -1), \tag{7}$$

where

$$G_{0\,2}^{1\,0}(\frac{p^2r^2}{4}|0,-1) = \frac{1}{2\pi\imath} \int_L dz \left(\frac{p^2r^2}{4}\right)^z \frac{\Gamma(-z)}{\Gamma(2+z)} \tag{8}$$

is the well-known Meijer function. The contour L is a loop starting and ending at $+\infty$ and encircling the poles of $\Gamma(-z)$ once in the negative direction. Since $p^2r^2/4$ is positive, one can show that the contour L can be distorted to become parallel to the imaginary axis and lying in the strip $-1/2 < Re(z) < 0$. Then substituting eq.(8) in eq.(7) and interchanging the order of integrations, we get

$$I_\delta^0(p^2) = \frac{1}{2\pi\imath} \int_{c-\imath\infty}^{c+\imath\infty} dz \left(\frac{p^2}{4}\right)^z \frac{\Gamma(-z)}{\Gamma(2+z)} \pi^2 \left(-\kappa\right)^{z-n+2} \gamma(n-z-2,-\kappa/\delta^2), \tag{9}$$

$$-1 < c < 0.$$

Here, $\gamma(\alpha, x)$ is the incomplete gamma function which can be expressed as

$$\gamma(\alpha, x) = x^\alpha \, \Gamma(\alpha) \, \gamma^*(\alpha, x) \, , \tag{10}$$

with $\gamma^*(\alpha, x)$ being an analytic function of α and x. Eq.(9) then becomes

$$I_\delta^0(p^2) = \frac{\pi^2 \delta^{4-2n}}{2\pi\imath} \int_{c-\imath\infty}^{c+\imath\infty} dz \left(\frac{p^2\delta^2}{4}\right)^z \frac{\Gamma(-z)\,\Gamma(n-2-z)}{\Gamma(2+z)} \gamma^*(n-z-2,-\kappa/\delta^2), \tag{11}$$

$$-1 < c < 0.$$

Notice, that after interchanging the order of integrations, the range of c can be extended. The imaginary part acquired by $I_\delta^0(p^2)$ after rotating p^2 to Minkowski space ($p^2 \to e^{-\imath\pi}p^2$), comes from the $\ln p^2$ terms in the series expansion of the right-hand side of eq.(11). These terms are produced by the double poles of the integrand in eq.(11) at $z = m$, $m = n-2, n-1, n, \ldots$. The result is

$$Im I_\delta^0(e^{-\imath\pi}p^2) = \pi^3 \delta^{4-2n} \sum_{m=n-2}^\infty \left(\frac{p^2\delta^2}{4}\right)^m \frac{(-1)^{m-n+2}}{m!\,(m+1)!\,(m-n+2)!} \times$$
$$\gamma^*(n-m-2,-\kappa/\delta^2). \tag{12}$$

Using eq.(10), one can show that

$$\gamma^*(-\alpha, x) = x^\alpha, \quad \alpha = 0, 1, 2, \ldots, \tag{13}$$

which implies

$$Im I_\delta^0(e^{-i\pi}p^2) = \pi^3 \kappa^{2-n} \sum_{m=n-2}^{\infty} \left(\frac{p^2\kappa}{4}\right)^m \frac{1}{m!\,(m+1)!\,(m-n+2)!}. \tag{14}$$

We now perform the integrals over the sizes of the instanton and the antiinstanton to get the well-known semiclassical result.

$$\sigma_{inc}^0 = \frac{\pi^3 C}{(\pi^2 v^2)^{4+2\alpha-n}} \sum_{m=n-2}^{\infty} \left(\frac{p^2}{4\pi^2 v^2}\right)^m \frac{[\Gamma(\alpha+m-n+3)]^2}{m!\,(m+1)!\,(m-n+2)!}. \tag{15}$$

It is important to note that the δ-dependence of the imaginary part of $I_\delta^0(e^{-i\pi}p^2)$ has completely disappeared in eq.(15). Consequently, σ_{inc}^0 is δ-independent for any $\delta > 0$ and its exponential growth with energy results only from the boundary at infinity of the Euclidean x-space in eq.(6).The virulent ultraviolate divergence, as well as the contribution of any "finite" part of the Euclidean x-space in eq.(6), do not seem to play any essential role.

III. THE LADDER GRAPHS.

We will now turn to the calculation of the Euclidean space ladder graphs shown in fig.2. The graphs, after rotating $p^2 \to e^{-i\pi} p^2$ and taking the imaginary part, constitute an important class of multiinstanton corrections to the leading semiclassical approximation of σ_{inc}. The forward scattering amplitude which corresponds to the ladder graph with $k+1$ rungs $(k = 0, 2, 4, \ldots)$ is given by

$$C^{k+1} \prod_{i=0}^{k} \prod_{j=1}^{k} \int_0^\infty d\rho_i^2\, \rho_i^{2\alpha}\, e^{-\pi^2 v^2 \rho_i^2} \int_0^\infty d\widetilde{\rho}_i^{\,2}\, \widetilde{\rho}_i^{\,2\alpha}\, e^{-\pi^2 v^2 \widetilde{\rho}_i^2} \times \tag{16}$$

$$\int d^4 q_j \frac{1}{q_j^2} \int d^4 x_i\, e^{-i(q_i + q_{i+1})x_i}\, F(x_i^2, \rho_i^2, \widetilde{\rho}_i^2) \quad,$$

where $q_0 = p_1$, $q_{k+1} = p_2$ and the four momenta q_j $(j = 1, 2, \ldots, k)$ are indicated in fig.2, $\rho_i, \widetilde{\rho}_i$ $(i = 0, 1, \ldots, k)$ are the scale sizes of the i-th instanton and the i-th antiinstanton respectively and x_i is their Euclidean separation.

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We will first consider the integrals over the instanton-antiinstanton separations

$$I_\delta^k(p^2) = \prod_{j=1}^k \int d^4q_j \, \frac{1}{q_j^2} \, \prod_{i=0}^k \int_{x_i^2 \geq \delta^2} d^4x_i \, e^{-i(q_i+q_{i+1})} \, F(x_i^2, \rho_i^2, \tilde{\rho}_i^2) \quad , \tag{17}$$

where the Euclidean space ultraviolet divergences are again regularized by restricting the x_i-integrations to $x_i^2 \geq \delta^2$. Repeating the analysis of the previous section, we obtain

$$I_\delta^k(p^2) = \frac{\pi^{2(k+1)} \, \delta^{(4-2n)(k+1)}}{(2\pi i)^{k+1}} \prod_{i=0}^k \int_{c_i-i\infty}^{c_i+i\infty} dz_i \left(\frac{\delta^2}{4}\right)^{z_i} \frac{\Gamma(-z_i) \, \Gamma(n-2-z_i)}{\Gamma(2+z_i)} \times$$

$$\gamma^*(n - z_i - 2, -\frac{\kappa_i}{\delta^2}) \, A_k(z_0, z_1, \ldots, z_k) \tag{18}$$

where $\kappa_i = \pi^2 \rho_i^2 \, \tilde{\rho}_i^2 \, v^2$ and (see fig.2)

$$A_k(z_0, z_1, \ldots, z_k) = \prod_{j=1}^k \int d^4q_j \, \frac{1}{q_j^2} \, \prod_{i=0}^k (q_i + q_{i+1})^{2z_i} \quad . \tag{19}$$

The constants c_i ($i = 0, 1, \ldots, k$) which satisfy the inequalities $-1 < c_i < 0$ (see eq.(11)) may and in fact will have to be further restricted in eq.(18) so that $A_k(z_0, z_1, \ldots, z_k)$ and the z_i -integrals exist. For the moment we just assume that there is some region of z_i 's in which $A_k(z_0, z_1, \ldots, z_k)$ exist and we restrict ourselves in this region. This assumption will be proved to be correct a posteriori (see below). The $s = (p_1 + p_2)^2$ dependence of A_k at $p_1^2 = p_2^2 = 0$ can be easily found from dimensional arguments (there are no infrared divergences)[1]. We get

$$A_k(z_0, z_1, \ldots, z_k) = F_k(z_i) \, s^{k+\sum_i z_i}. \tag{20}$$

It is easily seen that, for $p_1^2 = p_2^2 = 0$,we also have

$$p_2 \cdot \frac{\partial}{\partial p_2} A_k = s \frac{\partial}{\partial s} A_k, \tag{21}$$

and, thus,

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[1]This semieuristic argument can be further corroborated by an explicit although tedious calculation.

$$p_2 \cdot \frac{\partial}{\partial p_2} A_k = (k + \sum_i z_i) A_k = z_k A_k - z_k A_k^{q_k}(z_0, z_1, \ldots, z_k - 1). \qquad (22)$$

$A_k^{q_k}$ denotes the expression A_k with the q_k- propagator ommited. Eq.(22) then gives

$$A_k = -\frac{z_k}{k + \sum_{i \neq k} z_i} A_k^{q_k}(z_0, z_1, \ldots, z_k - 1) \ , \ \ k + \sum_{i \neq k} z_i \neq 0 \ . \qquad (23)$$

The q_k-integration in $A_k^{q_k}$ can now be performed :

$$\int d^4 q_k \, [(q_k + p_2)^2]^{z_k - 1} \, [(q_{k-1} + q_k)^2]^{z_{k-1}} =$$

$$\pi^2 \frac{\Gamma(-1 - z_k - z_{k-1})}{\Gamma(1 - z_k)\,\Gamma(-z_{k-1})} B(1 + z_k, 2 + z_{k-1}) \, [(q_{k-1} - p_2)^2]^{1 + z_k + z_{k-1}}, \qquad (24)$$

for $-1 < Re(z_k) < 1$, $-2 < Re(z_{k-1})$, $Re(z_k) + Re(z_{k-1}) < -1$. We then obtain the reccurence formula

$$A_k(z_0, z_1, \ldots, z_k) = \frac{\pi^2}{[k + \sum_{j \neq k} z_j]} \frac{\Gamma(-1 - z_k - z_{k-1})}{\Gamma(-z_k)\,\Gamma(-z_{k-1})} \times$$

$$B(1 + z_k, \, 2 + z_{k-1}) \, A_{k-1}(z_0, z_1, \ldots, z_{k-2}, \, 1 + z_k + z_{k-1}) \ , \qquad (25)$$

where $Re(z_k) < 0$ and A_k $(k = 1, 3, \ldots)$ is also defined by eq.(19) but with $q_{k+1} = -p_2$. Note that eq.(25) obviously holds for $k = 1, 3, 5, \ldots$ too. Introducing the function

$$D_k(z_0, z_1, \ldots, z_k) = \prod_{i=0}^{k} \frac{\Gamma(-z_i)}{\Gamma(2 + z_i)} A_k(z_0, z_1, \ldots, z_k) \quad k = 0, 1, 2, \ldots, \qquad (26)$$

the reccurence formula in eq.(25) takes the simple form

$$D_k(z_0, z_1, \ldots, z_k) = \frac{\pi^2}{(1 + z_k)(k + \sum_{j \neq k} z_j)} \, D_{k-1}(z_0, z_1, \ldots, z_{k-2}, 1 + z_{k-1} + z_k) \qquad (27)$$

and can be easily solved to give

$$D_k(z_0, z_1, \ldots, z_k) = \pi^{2k} \prod_{m=1}^{k} \frac{1}{(m + z_0 + \cdots + z_{m-1})(k + 1 - m + z_m + \cdots + z_k)}$$

$$\times \frac{\Gamma(-k - \sum_{i=0}^{k} z_i)}{\Gamma(2 + k + \sum_{i=0}^{k} z_i)} \, s^{k + \sum_i z_i}. \qquad (28)$$

This formula holds provided that

$$-2 < Re(z_m) \quad m = 0, 1, \ldots, k - 1 \ ;$$

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$$-1 < k - m + Re(z_m + \cdots + z_k) < 0 \quad m = 1, 2, \ldots, k) \ ;$$

$$k + Re(z_0 + \cdots + z_k) < 0 \ ;$$

$$m + z_0 + \cdots + z_{m-1} \neq 0 \quad m = 1, 2, \ldots, k \tag{29}$$

as can be easily deduced from the restrictions which follow eqs.(19), (24) and (25). Substituting eq(28) in eq.(18) we obtain

$$I_\delta^k(s) = \frac{4^k \pi^{2(2k+1)}}{(2\pi\imath)^{k+1}} \prod_{\imath=0}^{k} \int_{c_\imath - \imath\infty}^{c_\imath + \imath\infty} dz_\imath \, (\delta^2)^{2-n+z_\imath} \, \Gamma(n - 2 - z_\imath) \, \gamma^*(n - 2 - z_i, -\frac{\kappa_\imath}{\delta^2}) \times$$

$$\prod_{m=1}^{k} \frac{1}{(m + z_0 + \cdots + z_{m-1})(k + 1 - m + z_m + \cdots + z_k)} \times \tag{30}$$

$$\frac{\Gamma(-k - \sum_{\imath=0}^{k} z_\imath)}{\Gamma(2 + k + \sum_{\imath=0}^{k} z_\imath)} \left(\frac{s}{4}\right)^{k + \sum_\imath z_\imath}$$

with $-1 < c_\imath < 0 \quad (\imath = 0, 1, \ldots, k)$ and $k + c_0 + \cdots + c_k < 0$.

The $z_\imath$- integrals can be evaluated by collapsing their contours to the right and using residue calculus. Since we are only interested in the imaginary part acquired by the amplitude when $s \to e^{-\imath\pi}s$, we only keep contributions proportional to $\ln s$. The relevant contributions to the first k $z_\imath$-integrals ($\imath = 0, 1, \ldots, k - 1$) then come from the simple poles of the functions $\Gamma(n - 2 - z_i)$ ($\imath = 0, 1, \ldots, k - 1$) whereas the z_k-integral gets contributions from the double poles of the product $\Gamma(n - 2 - z_k)\Gamma(-k - \sum_{\imath=0}^{k} z_\imath)$ where the first k $z_\imath$'s ($\imath = 0, 1, \ldots, k - 1$) have already been substituted by integers. The final result is

$$Im \, I_\delta^k(se^{-\imath\pi}) = 4^k \pi^{4k+3} \sum_{n-1 \leq l_0, l_1, \ldots, l_k} \prod_{\imath=0}^{k} \frac{(\kappa_\imath)^{l_\imath + 1 - n}}{(l_\imath + 1 - n)!} \times$$

$$\prod_{m=1}^{k} \frac{1}{[l_0 + \cdots + l_{m-1}][l_m + \cdots + l_k]} \times \frac{(s/4)^{\sum_{\imath=0}^{k} l_\imath - 1}}{[\sum_{\imath=0}^{k} l_\imath - 1]! \, [\sum_{\imath=0}^{k} l_\imath]!} \tag{31}$$

and turns out to be again δ-independent. Performing the $\rho_\imath^2, \tilde{\rho}_\imath^2$ -integrals in eq.(16) we finally obtain the contribution of the ladder graph with $k + 1$ rungs ($k = 0, 2, 4, \ldots$) to σ_{inc} :

$$\sigma_{inc}^k = \frac{1}{\pi s} \left[4\pi^4 \, C \, (\pi^2 v^2)^{n-3-2\alpha} \right]^{k+1} \sum_{n-1 \leq l_0, l_1, \ldots, l_k} \prod_{\imath=0}^{k} \frac{[\Gamma(l_\imath + \alpha + 2 - n)]^2}{\Gamma(l_\imath + 2 - n)} \times$$

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$$\prod_{m=1}^{k} \frac{1}{[l_0 + \cdots + l_{m-1}][l_m + \cdots + l_k]} \times \frac{(s/4\pi^2 v^2)^{\sum_{i=0}^{k} l_i}}{[\sum_{i=0}^{k} l_i - 1]! \, [\sum_{i=0}^{k} l_i]!} \tag{32}$$

The multiple Series found for σ_{inc}^k, as it stands, looks very complicated to be handled. We shall attempt to get an estimate by finding suitable upper and lower bounds. In order to achieve this, we shall make extensive use of the inequalities

$$l_0 \, l_1 \cdots l_k \leq \frac{1}{(k+1)!} (l_0 + l_1 + \cdots + l_k)^{k+1} \quad ,$$

$$\frac{1}{l_0 \, l_1 \cdots l_k} > \frac{(k+1)!}{(l_0 + l_1 + \cdots + l_k)^{k+1}} . \tag{33}$$

Taking into account that $\alpha > 2$ and defining $a = [\alpha] + 1$ we find that

$$\sigma_l^k < \sigma_{inc}^k < \sigma_u^k , \tag{34}$$

where

$$\sigma_l^k = \frac{1}{\pi \, s} D^{k+1} \, 2^k \sum_{n-1 \leq l_0, l_1, \ldots, l_k} \frac{1}{\left[\sum_{i=0}^{k} l_i\right]^{n(k+1)-1}} \frac{\prod_{i=0}^{k} \Gamma(l_i)}{\left[\Gamma(\sum_{i=0}^{k} l_i)\right]^2} \left[\frac{s}{4\pi^2 v^2}\right]^{\sum l_i} ,$$

$$\sigma_u^k = \frac{1}{\pi \, s} D^{k+1} \frac{1}{(n-1)^{2k}} \frac{1}{(k+1)^{2a}} \frac{1}{(k!)^{2(a+1)}} \left(\frac{a^a}{a!}\right)^{2(k+1)} \times$$

$$\sum_{n-1 \leq l_0, l_1, \ldots, l_k} \left[\sum_{i=0}^{k} l_i\right]^{2a(k+1)-1} \frac{\prod_{i=0}^{k} \Gamma(l_i)}{\left[\Gamma(\sum_{i=0}^{k} l_i)\right]^2} \left[\frac{s}{4\pi^2 v^2}\right]^{\sum l_i} , \tag{35}$$

with $D = 4\pi^4 \, C \, (\pi^2 v^2)^{n-3-2\alpha}$.

It is now clear that the next step must be the study of the multiple Series

$$\Sigma_k(x) = \frac{1}{x} \sum_{l_i=1}^{\infty} \frac{\Gamma(l_0) \cdots \Gamma(l_k)}{[\Gamma(l_0 + l_1 + \cdots + l_k)]^2} x^{l_0 + l_1 + \cdots + l_k} , \quad x = \frac{s}{4\pi^2 v^2} . \tag{36}$$

Then, σ_l^k and σ_u^k could be recovered by integrating or differentiating $\Sigma_k(x)$ with respect to $\ln(x)$ a suitable number of times. The fact that the l_i-summations in the definition of σ_l^k and σ_u^k start at $n-1$ cannot change the results in any fundamental way.

The Laplace transform of $\Sigma_k(x)$ is

$$S_k(t) = \int_0^\infty dx \, e^{-xt} \Sigma_k(x) =$$

$$\sum_{l_i=1}^{\infty} \frac{\Gamma(l_0)\,\Gamma(l_1)\cdots\Gamma(l_k)}{\Gamma(l_0 + l_1 + \cdots + l_k)}\, t^{-(l_0 + l_1 + \cdots + l_k)} \tag{37}$$

We can now use the integral representation for the generalized beta function

$$\frac{\Gamma(l_0)\,\Gamma(l_1)\cdots\Gamma(l_k)}{\Gamma(l_0 + l_1 + \cdots + l_k)} = \int_0^1 \prod_{i=0}^k d\alpha_i \prod_{j=0}^k \alpha_j^{l_j - 1}\, \delta(1 - \textstyle\sum \alpha_i) \tag{38}$$

to get

$$S_k(t) = \int_0^1 \frac{\prod d\alpha_i}{\prod \alpha_i}\delta(1 - \textstyle\sum\alpha_i) \sum_{l_i=1}^{\infty} \left[\frac{\alpha_0}{t}\right]^{l_0} \cdots \left[\frac{\alpha_k}{t}\right]^{l_k}. \tag{39}$$

The summations are now decoupled and can be readily performed provided that $\mid \alpha_i/t \mid <$ 1. We obtain

$$S_k(t) = \int_0^1 \frac{\prod d\alpha_i\, \delta(1 - \sum\alpha_i)}{(t - \alpha_0)\,(t - \alpha_1)\cdots(t - \alpha_k)}\quad. \tag{40}$$

The inverse Laplace transform of the expression above can be found and the answer is

$$S_k(x) = \int_0^1 \prod d\alpha_i\, \delta(1 - \textstyle\sum\alpha_i) \sum_{m=0}^k \frac{e^{\alpha_m x}}{P_m(\alpha_m)} \tag{41}$$

where $P_m(y)$ is a polynomial given by

$$P_m(y) = \frac{\prod_{i=0}^k (y - \alpha_i)}{y - \alpha_m}\quad. \tag{42}$$

Now the α_i- integrations can be performed and we end up with the following recursive formula

$$\Sigma_k(x) = e^{\frac{x}{k}} \int_0^x dz\, e^{-\frac{z}{k}} \int_0^1 d\rho\, \Sigma_{k-1}(\rho\, z)\quad. \tag{43}$$

This integral equation can be transformed into an integrodifferential equation

$$\frac{d}{dx}\Sigma_k(x) = \frac{1}{k}\,\Sigma_k(x) + \frac{1}{x}\int_0^x \Sigma_{k-1}(z)\, dz \tag{44}$$

or a differential equation

$$x\frac{d^2}{dx^2}f_k(x) + \left[\left(2 - \frac{1}{k}\right)x + 1\right]\frac{d}{dx}f_k(x) + \left(1 - \frac{1}{k}\right)(x + 1)\, f_k(x) = f_{k-1}(x), \tag{45}$$

where

$$f_k(x) = e^{-x}\, \Sigma_k(x) \cdot \tag{46}$$

When x is large, the differential equation reduces to

$$\frac{d^2}{dx^2} f_k(x) + \left[2 - \frac{1}{k}\right] \frac{d}{dx} f_k(x) + \left[1 - \frac{1}{k}\right] f_k(x) = \frac{f_{k-1}(x)}{x} \quad . \tag{47}$$

One particular integral of this equation is

$$f_k(x) = \frac{k}{k-1} \frac{1}{1 + \frac{d}{dx}} \frac{1}{1 + \frac{k}{k-1}\frac{d}{dx}} \frac{f_{k-1}(x)}{x} \tag{48}$$

with $f_1(x) = 1$.

It is now obvious that the leading order solution of eq.(45) can be written as

$$f_k(x) = \frac{k}{x^{k-1}} \tag{49}$$

leading to

$$\Sigma_k(x) = k\, \frac{e^x}{x^{k-1}} \cdot \tag{50}$$

We have already pointed out that the upper and lower bounds for σ_{inc}^k which were defined in eq.(35) can be related to derivatives or integrals of $\Sigma_k(x)$ with respect to $\ln x$. Such operations, however, cannot modify the form of $\Sigma_k(x)$ in an essential way since the exponential growth cannot be affected . We do expect a change in the leading power of x and of course the constant coefficient will be different. We conclude that $\sigma_l^k \sim c_l^k\, e^x\, x^{-m_1}$, $\sigma_u^k \sim c_u^k\, e^x\, x^{-m_2}$ where the constants c_l^k, c_u^k , m_1 , m_2 can in principle be calculated. The cross section σ_{inc}^k being bound between two exponentials, can only behave exponentially, possibly modified by an asymptotic Series of inverse powers of s . Taking into account that σ_{inc}^k for $k > 1$ is highly supressed by the small D^{k+1} factor containing the instanton 't Hooft factor, we deduce that the contribution of all ladder graphs for $k > 1$ is negligible and cannot alter the high energy behavior of the leading order result. In particular they do not affect the possible validity of the ZMS picture, that is based on the instanton-antiinstanton chain graphs only.

This result is not totally unexpected. It is known that to each shaded blob of fig.2, which represents exchange of any number of bosons and $4n_g - 2$ fermions between an instanton

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and an intiinstanton, corresponds an exponentially growing factor, while, to each instanton or antiinstanton, an exponentially small 't Hooft factor. Since the number of instanton or antiinstanton vertices outnumbers the number of the multiparticle-exchange blobs by a factor of two, we can expect that the contribution of such ladders is supressed compared to the leading semiclassical result. Moreover, the fact that not all the momentum flows through each rung makes at the end the ladder graphs grow only exponentially with s. This is in contrast to the case of a linear chain where all momentum flows between the instanton and the antiinstanton, creating thus an exponential growth that can counterbalance the suppression effect of the 't Hooft factors.

REFERENCES

[1] G.'t Hooft, Phys.Rev.Lett **37** (1976) 8; Phys.Rev. **D14** (1976) 3432.

[2] N.H. Christ, Phys.Rev **D21** (1980) 1951; H. Aoyama and H. Goldberg, Phys.Lett **188B** (1987) 506; A. Ringwald, Nucl.Phys. **B330** (1989) 1; O. Espinosa, Nucl.Phys. **B343** (1990) 310.

[3] M. Porrati, Nucl.Phys. **B347** (1990) 371; P. Arnold and M. Mattis, Phys.Rev. **D44** (1991) 3650.

[4] N.S. Manton, Phys.Rev. **D28** (1983) 2019; F.R. Klinkhammer and N.S. Manton, Phys.Rev. **D30** (1984) 2212.

[5] H. Aoyama and H. Kikkuchi, Phys.Lett. **B247** (1990) 75; Phys.Rev. **D43** (1991) 1999; V. Zakharov, Nucl.Phys. **B353** (1991) 683; M. Maggiore and M. Shifman, Nucl.Phys. **B371** (1992) 177.

[6] N. Dorey and M.P. Mattis, Phys.Lett. **B277** (1992) 337.

[7] V.V. Khoze and A. Ringwald, Nucl.Phys. **B355** (1991) 351; **B259** (1991) 106.

[8] A.H. Mueller, Nucl.Phys. **B353** (1991) 44; S.Yu.Khlebnikov, V.A. Rubakov, P.G. Tinyakov, "The lowest Correction to the Total Instanton Induced Cross Section in the Electroweak Theory below 10 TeV", INR preprint (Moscow,1991).

FIGURE CAPTIONS

Fig.1 . The graph which correponds to the leading semiclassical approximation to the forward $2 \to 2$ scattering amplitude. Single lines represent fermions, the filled and blank circles represent instanton and antiinstanton vertices respectively, whereas the shaded blob represents the exchange of $4n_g - 2$ fermions and any number of Higgs bosons. n_g is the number of fermion generations.

Fig.2 . The ladder graph with $k + 1$ rungs $(k = 0, 2, 4 \ldots)$. Notation as in fig. 1. The complex parameters $z_\imath$ $(\imath = 0, 1, \ldots, k)$ which appear in eq.(18) are also indicated.
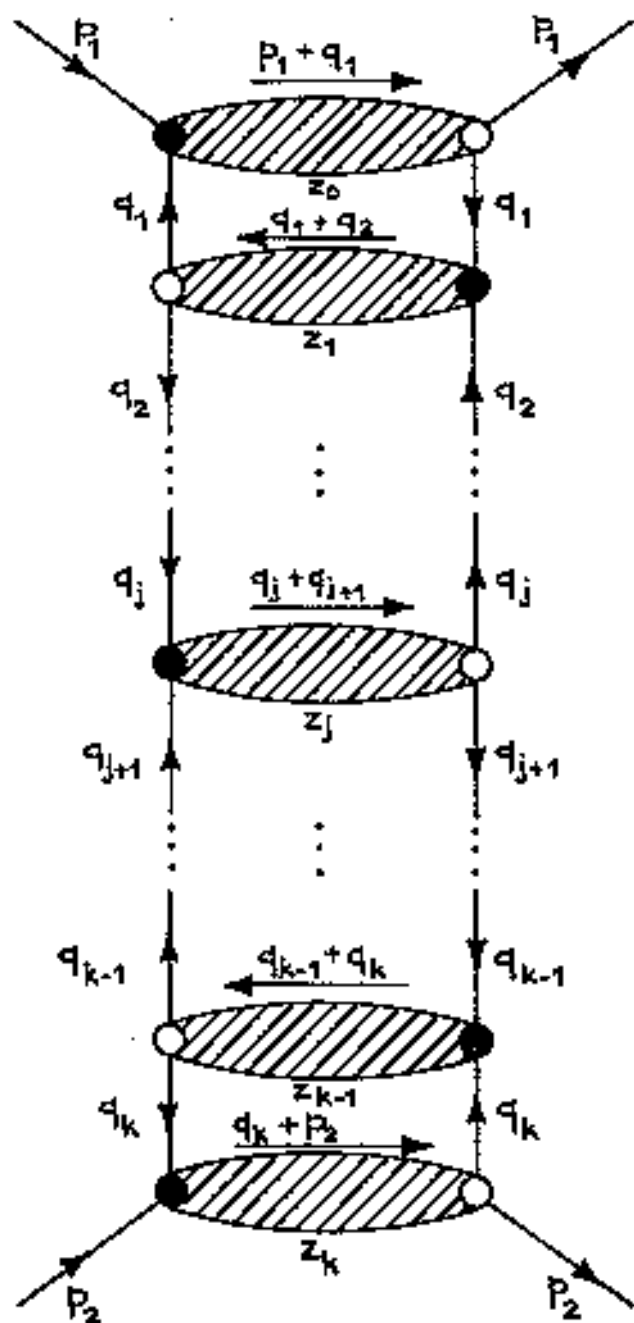
Figure 1.

Figure 2.